Déterminant - Approfondissement

Exercice 1

On considère les matrices $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ et $B = \begin{pmatrix} 2 & 4 \\ 5 & 1 \end{pmatrix}$.

1. On a : $det(A) = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 1 \times 3 - 2 \times (-1) = 5 \neq 0$. A est inversible.

   On a : $det(B) = \begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = 2 \times 1 - 4 \times (5) = -18 \neq 0$. B est inversible.

2. $det(3A) = \begin{vmatrix} 3 & 6 \\ -3 & 9 \end{vmatrix} = 3 \times 9 - 6 \times (-3) = 45 = 9 \times 5 = 9 \times det(A) = 3^2 \times det(A)$

   $det(-2B) = \begin{vmatrix} -4 & -8 \\ -10 & -2 \end{vmatrix} = -4 \times (-2) - (-8) \times (-10) = -72 = 4 \times (-18) = (-2)^2 \times det(B)$

3. AB = $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 6 \\ 13 & -1 \end{pmatrix}$ et BA = $\begin{pmatrix} 2 & 4 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 16 \\ 4 & 13 \end{pmatrix}$.

   On note que AB ≠ BA.

4. $det(AB) = \begin{vmatrix} 12 & 6 \\ 13 & -1 \end{vmatrix} = 12 \times (-1) - 6 \times (13) = -90 = 5 \times (-18) = det(A) \times det(B)$.

   $det(BA) = \begin{vmatrix} -2 & 16 \\ 4 & 13 \end{vmatrix} = -2 \times (13) - 16 \times (4) = -90 = 5 \times (-18) = det(A) \times det(B)$.

5. Conclusion : $det(AB) = det(BA) = det(A) \times det(B)$.

Exercice 2

On considère la matrice carrée $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

On appelle déterminant de la matrice A , ce que l'on note det(A), l'expression : ad - bc.

Soit la matrice carrée $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ et λ un nombre réel.

1. Démontrons que det(λA) = $\lambda^2 \times$ det(A).

   $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, donc : $\lambda A = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$.

   D'où : $det(\lambda A) = \begin{vmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{vmatrix} = \lambda a \times \lambda d - \lambda b \times (\lambda c) = \lambda^2(ad - bc) = \lambda^2 det(A)$.

2. Démontrons que det(AB) = det(A) × det(B).

On a : AB = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}$.

Donc : $det(AB) = \begin{vmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{vmatrix} = (aa' + bc')(cb' + dd') - (ab' + bd')(ca' + dc')$

$= aa'cb' + aa'dd' + bc'cb' + bc'dd' - ab'ca' - ab'dc' - bd'ca' - bd'dc'$

$= aa'dd' + bcb'c' - adb'c' - bca'd'$

De plus : $det(A) \times det(B) = (ad - bc)(a'd' - b'c') = aa'dd' + bcb'c' - adb'c' - bca'd'$

D'où : det(AB) = det(A) × det(B).

3. Montrons que det(AB) = det(BA).

   On a : det(AB) = det(A) × det(B)

   Donc : det(BA) = det(B) × det(A).

   En résultat : det(AB) = det(A) × det(B) = det(B) × det(A) = det(BA)