# NUMBER THEORY

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## 1–1 PRINCIPLE OF MATHEMATICAL INDUCTION

Let us try to answer the following question: What is the sum of all integers from one through $n$, for any positive integer $n$? If $n = 1$, the sum equals 1 because 1 is the only summand. The answer we seek is a formula that will enable us to determine this sum for each value of $n$ without having to add the summands.

Table 1–1 lists the sum $S_n$ of the first $n$ consecutive positive integers for values of $n$ from 1 through 10. Notice that in each case $S_n$ equals one-half the product of $n$ and the next integer; that is,

$$S_n = \frac{n(n+1)}{2} \qquad (1\text{-}1\text{-}1)$$

for $n = 1,2,3,\ldots,10$. Although this formula gives the correct value of $S_n$ for the first ten values of $n$, we cannot be sure that it holds for $n$ greater than 10.

To construct Table 1–1, we do not need to compute $S_n$ each time by adding the first $n$ positive integers. Having obtained values of $S_n$

TABLE 1–1: SUM $S_n$ OF THE FIRST $n$ CONSECUTIVE POSITIVE INTEGERS.

| $n$ | $S_n$ | $n$ | $S_n$ |
|---|---|---|---|
| 1 | 1 | 6 | 21 |
| 2 | 3 | 7 | 28 |
| 3 | 6 | 8 | 36 |
| 4 | 10 | 9 | 45 |
| 5 | 15 | 10 | 55 |

for $n$ less than or equal to some integer $k$, we can determine $S_{k+1}$ simply by adding $(k+1)$ to $S_k$:

$$S_{k+1} = S_k + (k+1).$$

This last approach suggests a way of verifying equation (1–1–1). Suppose we know that formula (1–1–1) is true for $n \leq k$, where $k$ is a positive integer. Then we know that

$$S_k = \frac{k(k+1)}{2},$$

and so

$$\begin{aligned} S_{k+1} &= S_k + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \left(\frac{k}{2}+1\right)(k+1) \\ &= \frac{(k+2)(k+1)}{2}, \end{aligned}$$

that is,

$$S_{k+1} = \frac{(k+1)((k+1)+1)}{2}.$$

The last equation is the same as equation (1–1–1) except that $n$ is replaced by $k+1$.

We have proved that if equation (1–1–1) holds for $n \leq k$, then it holds for $n = k+1$, and we have already verified that equation (1–1–1) holds for $n = 1,2, \ldots, 10$. Therefore, by the preceding argument, we conclude that equation (1–1–1) is also correct for $n = 11$. Since it holds for $n = 1,2, \ldots, 11$, the same process shows that it is correct for $n = 12$. Since it is true for $n = 1,2, \ldots, 12$, it is true for $n = 13$, and so on. We can describe the principle underlying the foregoing argument in various ways. The following formulation is the most appropriate for our purposes.

**Principle of mathematical induction:** *A statement about integers is true for all integers greater than or equal to 1 if*

(i) *it is true for the integer 1, and*

(ii) *whenever it is true for all the integers $1,2,\ldots,k$, then it is true for the integer $k+1$.*

By "a statement about integers" we do not necessarily mean a formula. A sentence such as "$n(n^2-1)(3n+2)$ is divisible by 24" is also





acceptable (see Exercise 17 of this section). The assumption that "the statement is true for $n=1,2,\ldots,k$" will often be referred to as the *induction hypothesis*. Sometimes the role 1 plays in the Principle will be replaced by some other integer, say $b$; in such instances the principle of mathematical induction establishes the statement for all integers $n \geq b$.